

Noncommutative gravity at second order via Seiberg-Witten map

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Abstract

We develop a general strategy to express noncommutative actions in terms of commutative ones by using a recently developed geometric generalization of the Seiberg-Witten map (SW map) between noncommutative and commutative fields.

We apply this general scheme to the noncommutative vierbein gravity action and provide a SW differential equation for the action itself as well as a recursive solution at all orders in the noncommutativity parameter θ . We thus express the action at order θ^{n+2} in terms of noncommutative fields of order at most θ^{n+1} and, iterating the procedure, in terms of noncommutative fields of order at most θ^n .

This in particular provides the explicit expression of the action at order θ^2 in terms of the usual commutative spin connection and vierbein fields. The result is an extended gravity action on commutative spacetime that is manifestly invariant under local Lorentz rotations and general coordinate transformations.

1 Introduction

Generalizations and extensions of Einstein gravity have a long history, with motivations that are both theoretical and experimental, see for example the reviews in [1]. In the last decade numerous higher order extensions of the Einstein action have been considered, mostly for applications in cosmological phenomenology, related to the issues of dark matter and dark energy.

A recent way to obtain higher order gravity theories is based on noncommutativity of spacetime. Gravity on noncommutative spacetime may be expected to capture some aspects of quantum gravity since there are indications that at short distances spacetime indeed becomes noncommutative (see for example [2] and references therein). Thus an extended higher order gravity theory on commutative spacetime, obtained from gravity on noncommutative spacetime, could be seen as an effective theory of a more fundamental quantum theory.

We present a manifestly Lorentz invariant expression for the first nontrivial term of the extended gravity theory obtained from the noncommutative vierbein gravity studied in [3, 4]. The resulting action is geometric and hence invariant under general coordinate transformations. It predicts specific higher curvature couplings, whose theoretical and phenomenological properties are ripe for being investigated and compared with other extended gravity approaches.

★-Product noncommutativity

Commutative spacetime can be described via the commutative algebra of complex valued functions on spacetime. Noncommutative spacetime can be described by a noncommutative deformation of this algebra. One way of deforming the commutative product of functions is by means of a \star -product. We consider \star -products originating from twist deformation: in this case the deformation depends on a dimensionful parameter θ that is a constant antisymmetric matrix with components θ^{AB} , and on a set of commuting vector fields X^A . It generalizes the Moyal-Groenewold [5] \star -product between phase-space functions. The \star -product between functions (i.e. 0-forms), and in general between arbitrary exterior forms (the \star -deformed exterior product), is defined by :

$$\begin{aligned} \tau \wedge_{\star} \tau' &\equiv \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n \theta^{A_1 B_1} \dots \theta^{A_n B_n} (\ell_{X_{A_1}} \dots \ell_{X_{A_n}} \tau) \wedge (\ell_{X_{B_1}} \dots \ell_{X_{B_n}} \tau') \\ &= \tau \wedge \tau' + \frac{i}{2} \theta^{AB} (\ell_{X_A} \tau) \wedge (\ell_{X_B} \tau') + \frac{1}{2!} \left(\frac{i}{2}\right)^2 \theta^{A_1 B_1} \theta^{A_2 B_2} (\ell_{X_{A_1}} \ell_{X_{A_2}} \tau) \wedge (\ell_{X_{B_1}} \ell_{X_{B_2}} \tau') + \dots \end{aligned} \quad (1.1)$$

where ℓ_{X_A} are Lie derivatives along commuting vector fields X_A . This product is noncommutative in the regions of spacetime where the vector fields X_A are nonvanishing, and is associative due to $[X_A, X_B] = 0$.

If spacetime is flat Minkowski space and the vector fields X_A are chosen to coincide with the partial derivatives ∂_{μ} , and if τ, τ' are 0-forms, then $\tau \star \tau'$ reduces to the well-known Moyal-Groenewold product [5]. In particular for coordinate functions we have $[x^{\mu}, x^{\nu}]_{\star} \equiv x^{\mu} \star x^{\nu} - x^{\nu} \star x^{\mu} = i\theta^{\mu\nu}$ (i.e. we have constant noncommutativity, since θ is constant). The more general \star -product (1.1) is given by a choice of spacetime dependent and commuting vector

fields $X_A = X_A^\mu(x)\partial_\mu$ that may be vanishing outside a given region (we refer to [6] Sec. 2.3 and [7] Sec. 5.2 for examples). In this case noncommutativity of the coordinates is spacetime dependent: $[x^\mu, x^\nu]_\star = i\theta^{AB}X_A^\mu(x)X_B^\nu(x) \equiv i\Theta^{\mu\nu}(x)$. Further properties of \star -products originating from abelian twists $\mathcal{F} = e^{-\frac{i}{2}\theta^{AB}X_A\otimes X_B}$, and of the associated twist differential geometry are summarized for example in Appendix A of [4].

\star -Gauge theory and Seiberg-Witten map (SW map)

Noncommutative gauge theories are obtained from nonabelian ones on commutative space by substituting ordinary products between fields and forms with \star -products. This recipe yields NC actions invariant under deformations of the original gauge symmetry, but brings into play many more fields than those present in the original action. This is easily understood: consider for example the \star -Yang-Mills field strength:

$$F_{\mu\nu}^I T_I = \partial_\mu A_\nu^I T_I - \partial_\nu A_\mu^I T_I - (A_\mu^J \star A_\nu^K - A_\nu^J \star A_\mu^K) T_J T_K . \quad (1.2)$$

Because of noncommutativity of the \star -product, anticommutators as well as commutators of group generators are appearing in the right-hand side. Since $T_J T_K$ must be a linear combination of T_I 's, we see we cannot in general consider the generators T_I to be a basis of the Lie algebra of the gauge group G . We have to enlarge the original set of Lie algebra generators and to consider the generators T_I to be a basis for the whole universal enveloping algebra of G . The range of the index I increases, and so does the number of independent field components A_μ^I . One can reduce this proliferation by choosing a specific representation for the generators T_I (for $SU(2)$ we can take its generators to be the Pauli matrices, so that a basis for the enveloping algebra only requires an additional matrix proportional to the unit matrix).

Even the remaining extra degrees of freedom can be eliminated, by use of the Seiberg-Witten map (SW map), relating the enveloping algebra valued gauge potential $A_\mu^I T_I$ to the original Lie algebra valued potential of the undeformed theory, the so-called classical gauge fields. The SW map applies also to matter fields, so that for example noncommutative matter fields in the adjoint $\Phi^I T_I$ are expressed in terms of the fewer Lie algebra valued commutative ones.

The SW map [8] was initially developed for $U(N)$ gauge fields and Moyal-Groenewold \star -products (constant noncommutativity), then applied to gauge fields of arbitrary gauge groups and extended to matter fields in [9, 10, 11]. An explicit solution for $U(1)$ gauge theory was presented in [12] developing earlier work [13, 14, 15]. Concerning nonabelian gauge groups an iterative procedure, based on recurrence relations, was devised in [16] (improving on results of [17], Sec. 6); it allows to construct the SW map as a power series expansion in θ in the particular case of Moyal-Groenewold \star -product.

The SW map is also mathematically very rich and was obtained as a power series in θ expansion for $U(1)$ gauge fields in [15] in case of an arbitrary \star -product (originating from an arbitrary Poisson tensor, i.e., nonconstant noncommutativity) using Kontsevich's results [18]; in the nonabelian case the situation (for nonconstant noncommutativity) is more involved and there is no definite result (despite interesting partial ones [19]).

In [4] we gave a geometric formulation of SW map and generalized the recurrence relations presented in [16] to the case of \star -products obtained via a set of mutually commuting vector fields.

These vector fields can be spacetime dependent and hence we obtained a SW map for nonabelian gauge fields with nonconstant noncommutativity on arbitrary (spacetime) manifolds.

It is now conceptually easy to obtain commutative actions from noncommutative ones: *i)* Consider the noncommutative gauge and matter fields as dependent on the commutative ones via the SW map; *ii)* Expand the star products and the NC fields in power series of the noncommutativity parameter θ .

The result is an action that contains higher order corrections in the field strength, its derivatives and derivatives of the matter fields, organized in a power series in θ . Every power in θ is separately invariant under ordinary gauge transformations, since the NC action is invariant under NC gauge transformations, and these latter are induced, via the SW map, by ordinary gauge transformations (that do not depend on θ) on the classical fields.

We have applied this strategy to vierbein gravity (where the gauge group is the local Lorentz group) coupled to fermions [4] (reviewed in [20]), to gauge fields [21] and to scalars [22]. In ref. [4] the second order correction (in θ) to pure vierbein gravity was presented in terms of first order fields. Here we give it in terms of classical fields and in a manifestly Lorentz gauge invariant form.

Plan of the paper

In Section 2 we recall the geometric action for pure NC vierbein gravity. Section 3 deals with the geometric SW map for \star -products with commuting vector fields (\star -products from abelian twists). In this section we consider an arbitrary gauge theory with nonabelian gauge group. We recall the SW differential equations and the associated recursive relations expressing the fields at order θ^{n+1} in terms of fields of order at most θ^n . Techniques for the calculation of the SW differential equation and recursive relations for composite fields are then provided.

In Section 4 we apply these results to the noncommutative vierbein gravity action. We establish the SW differential equation it satisfies and express the action at order θ^{n+2} in terms of the noncommutative spin connection and matter fields up to order θ^{n+1} , and, iterating the procedure, in terms of the fields up to order θ^n . In Section 5 we present the θ^2 correction of NC vierbein gravity, and of the NC cosmological term, in terms of classical fields.

In the appendices we list the Cartan formulae used throughout the main text and we summarize the $D = 4$ gamma matrix conventions.

2 NC vierbein gravity action

The noncommutative action reads [3]:

$$S_{NC} = \int Tr(i\gamma_5 \widehat{R} \wedge_\star \widehat{V} \wedge_\star \widehat{V}) , \quad (2.1)$$

where the curvature $\widehat{R}(\widehat{\Omega})$ is defined in terms of the NC spin connection as

$$\widehat{R} = d\widehat{\Omega} - \widehat{\Omega} \wedge_\star \widehat{\Omega} . \quad (2.2)$$

This definition implies the Bianchi identity:

$$D\widehat{R} \equiv d\widehat{R} - \widehat{\Omega} \wedge_\star \widehat{R} + \widehat{R} \wedge_\star \widehat{\Omega} = 0 . \quad (2.3)$$

The NC vierbein \widehat{V} , spin connection $\widehat{\Omega}$ and curvature \widehat{R} are valued in Dirac gamma matrices and the trace in the above action is taken on their spinor indices. The Dirac gamma matrices expansion of the NC fields is [23]:

$$\widehat{\Omega} = \frac{1}{4}\widehat{\omega}^{ab}\gamma_{ab} + i\widehat{\omega}1 + \widehat{\omega}\gamma_5 , \quad (2.4)$$

$$\widehat{V} = \widehat{V}^a\gamma_a + \widehat{\widetilde{V}}^a\gamma_a\gamma_5 , \quad (2.5)$$

$$\widehat{R} = \frac{1}{4}\widehat{R}^{ab}\gamma_{ab} + i\widehat{R} + \widehat{\widetilde{R}}\gamma_5 . \quad (2.6)$$

The classical limits of the NC fields are constrained to be the classical fields [3, 24, 4]:

$$\Omega \equiv \frac{1}{4}\omega^{ab}\gamma_{ab} , \quad V \equiv V^a\gamma_a , \quad R \equiv \frac{1}{4}R^{ab}\gamma_{ab} \quad (2.7)$$

with

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \quad (2.8)$$

where repeated Lorentz indices are summed with the Minkowski metric η_{ab} . By recalling that $Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5) = -4i\varepsilon_{abcd}$ one can easily check that the classical limit of the NC action (2.1) (obtained by replacing NC fields by classical fields, and deformed exterior products by ordinary exterior products) reproduces the usual first order formulation of the Einstein-Hilbert action:

$$S_{commutative} = \int Tr(i\gamma_5 R \wedge V \wedge V) = \int R^{ab} \wedge V^c \wedge V^d \varepsilon_{abcd} . \quad (2.9)$$

2.1 Noncommutative symmetries

The NC action (2.1) is invariant under general coordinate transformations (being the integral of a 4-form) and under the \star -gauge variations:

$$\begin{aligned} \widehat{\delta}_\varepsilon \widehat{\Omega} &= d\widehat{\varepsilon} - \widehat{\Omega} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{\Omega} \implies \widehat{\delta}_\varepsilon \widehat{R} = -\widehat{R} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{R} , \\ \widehat{\delta}_\varepsilon \widehat{V} &= -\widehat{V} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{V} \end{aligned} \quad (2.10)$$

with an arbitrary parameter $\widehat{\varepsilon}(x)$ commuting with γ_5 , i.e., valued in the even gamma matrix algebra,

$$\widehat{\varepsilon} = \frac{1}{4}\widehat{\varepsilon}^{ab}\gamma_{ab} + i\widehat{\varepsilon} + \widehat{\widetilde{\varepsilon}}\gamma_5 . \quad (2.11)$$

The invariance of the noncommutative action under these transformations relies on the cyclicity of the integral (and of the trace) and on $\widehat{\varepsilon}$ commuting with γ_5 . The classical limit of the gauge parameter is constrained to be:

$$\varepsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} \quad (2.12)$$

and one can check that the classical limit of the gauge transformations (2.10) reproduces the usual local Lorentz rotations on the vielbein and on the spin connection.

As discussed in the introduction, the extra fields entering in the expansions of $\widehat{\Omega}$, \widehat{V} and $\widehat{\varepsilon}$ are due to the noncommutativity of the star product. Indeed the \star -gauge variations of the fields (2.10) include also anticommutators of gamma matrices. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5 , we see that the corresponding fields must be included in the expansion of $\widehat{\Omega}$. Similarly, \widehat{V} must contain a $\gamma_a \gamma_5$ term due to $\{\gamma_{ab}, \gamma_c\}$.

All the components along the $SO(1, 3)$ enveloping algebra generators are taken to be real, and therefore fields and curvatures satisfy the hermiticity properties:

$$\widehat{\Omega}^\dagger = -\gamma_0 \widehat{\Omega} \gamma_0, \quad \widehat{V}^\dagger = \gamma_0 \widehat{V} \gamma_0, \quad \widehat{R}^\dagger = -\gamma_0 \widehat{R} \gamma_0, \quad (2.13)$$

i.e., $\widehat{\Omega}$ and \widehat{R} are γ_0 -antihermitian, while \widehat{V} is γ_0 -hermitian. Using these rules it is a quick matter to check that the noncommutative action (2.1) is real.

3 Geometric Seiberg-Witten map

The Seiberg-Witten map relates the noncommutative gauge field $\widehat{\Omega}$ to the ordinary Ω , and the noncommutative gauge parameter $\widehat{\varepsilon}$ to the ordinary ε so as to satisfy:

$$\widehat{\Omega}(\Omega) + \delta_{\widehat{\varepsilon}} \widehat{\Omega}(\Omega) = \widehat{\Omega}(\Omega + \delta_{\varepsilon} \Omega) \quad (3.1)$$

with

$$\delta_{\varepsilon} \Omega_{\mu} = \partial_{\mu} \varepsilon + \varepsilon \Omega_{\mu} - \Omega_{\mu} \varepsilon, \quad (3.2)$$

$$\delta_{\widehat{\varepsilon}} \widehat{\Omega}_{\mu} = \partial_{\mu} \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{\Omega}_{\mu} - \widehat{\Omega}_{\mu} \star \widehat{\varepsilon}. \quad (3.3)$$

In words: the dependence of the noncommutative gauge field on the ordinary one is fixed by requiring that ordinary gauge variations of Ω inside $\widehat{\Omega}(\Omega)$ produce the noncommutative gauge variation of $\widehat{\Omega}$.

Similarly noncommutative matter fields are related to the commutative ones by requiring

$$\widehat{\Phi}(\Phi, \Omega) + \delta_{\widehat{\varepsilon}} \widehat{\Phi}(\Phi, \Omega) = \widehat{\Phi}(\Phi + \delta_{\varepsilon} \Phi, \Omega + \delta_{\varepsilon} \Omega). \quad (3.4)$$

The conditions (3.1), (3.4) are implied by the following differential equations in the non-commutativity parameter θ^{AB} [8, 4]:

$$\frac{\partial}{\partial \theta^{AB}} \widehat{\Omega} = \frac{i}{4} \{ \widehat{\Omega}_{[A}, \ell_{B]} \widehat{\Omega} + \widehat{R}_{B]} \} \star, \quad (3.5)$$

$$\frac{\partial}{\partial \theta^{AB}} \widehat{\Phi} = \frac{i}{4} \{ \widehat{\Omega}_{[A}, \mathbb{L}_{B]} \widehat{\Phi} \} \star, \quad (3.6)$$

$$\frac{\partial}{\partial \theta^{AB}} \widehat{\varepsilon} = \frac{i}{4} \{ \widehat{\Omega}_{[A}, \ell_{B]} \widehat{\varepsilon} \} \star, \quad (3.7)$$

where:

- The bracket $[A B]$ denotes that the indices A and B are antisymmetrized with weight 1, so that for example $\widehat{\Omega}_{[A} \widehat{R}_{B]} = \frac{1}{2}(\widehat{\Omega}_A \widehat{R}_B - \widehat{\Omega}_B \widehat{R}_A)$. The bracket $\{ , \}_\star$ is the usual \star -anticommutator, for example $\{\Omega_A, R_B\}_\star = \Omega_A \star R_B + R_B \star \Omega_A$.
- ℓ_B is the Lie derivative along the vector field X_B .
- $\widehat{\Omega}_A, \widehat{R}_A$ are defined as the contraction i_A along the tangent vector X_A of the exterior forms $\widehat{\Omega}, \widehat{R}$, i.e. $\widehat{\Omega}_A \equiv i_A \widehat{\Omega}, \widehat{R}_A \equiv i_A \widehat{R}$.
- The second differential equation holds for fields transforming in the adjoint representation. Notice that $\widehat{\Phi}$ can also be an exterior form. The notation \mathbb{L}_B is defined by $\mathbb{L}_B \equiv \ell_B + L_B$ where L_B is the covariant Lie derivative along the tangent vector X_B ; it acts on the field $\widehat{\Phi}$ as

$$L_B \widehat{\Phi} = \ell_B \widehat{\Phi} - [\widehat{\Omega}_B, \widehat{\Phi}]_\star ,$$

where $[\widehat{\Omega}_B, \widehat{\Phi}]_\star = \widehat{\Omega}_B \star \widehat{\Phi} - \widehat{\Phi} \star \widehat{\Omega}_B$. In fact the covariant Lie derivative L_B can be written in Cartan form:

$$L_B = i_B D + D i_B , \quad (3.8)$$

where D is the covariant derivative.

The differential equations (3.5)-(3.7) hold for any abelian twist defined by arbitrary commuting vector fields X_A (that can vanish in some region of spacetime) [4]. They reduce to the usual Seiberg-Witten differential equations formulae [8] for an arbitrary nonabelian gauge group in case the twist becomes a Moyal-Groenewold twist, i.e., $X_A \rightarrow \partial_\mu$.

We solve these differential equations order by order in θ by expanding $\widehat{\Omega}$ and $\widehat{\Phi}$ in power series of θ

$$\widehat{\Omega} = \Omega^0 + \Omega^1 + \Omega^2 \dots + \Omega^n \dots , \quad \widehat{\Phi} = \Phi^0 + \Phi^1 + \Phi^2 \dots + \Phi^n \dots , \quad (3.9)$$

where the fields Ω^n and Φ^n are homogeneous polynomials in θ of order n . If we multiply the differential equations by θ^{AB} and use the identities $\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\Omega}^{n+1} = (n+1) \widehat{\Omega}^{n+1}$ and $\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\Phi}^{n+1} = (n+1) \widehat{\Phi}^{n+1}$, we obtain the recursive relations

$$\widehat{\Omega}^{n+1} = \frac{i \theta^{AB}}{4(n+1)} \{ \widehat{\Omega}_A, \ell_B \widehat{\Omega} + \widehat{R}_B \}_\star^n , \quad (3.10)$$

$$\widehat{\Phi}^{n+1} = \frac{i \theta^{AB}}{4(n+1)} \{ \widehat{\Omega}_A, \mathbb{L}_B \widehat{\Phi} \}_\star^n , \quad (3.11)$$

$$\widehat{\varepsilon}^{n+1} = \frac{i \theta^{AB}}{4(n+1)} \{ \widehat{\Omega}_A, \ell_B \widehat{\varepsilon} \}_\star^n , \quad (3.12)$$

where for any field P (also composite like for ex. $\{ \widehat{\Omega}_A, \mathbb{L}_B \widehat{\Phi} \}_\star$), P^{n+1} denotes its component of order $n+1$ in θ .

3.1 Differential equation and recursive relations for composite fields

The θ expansion of NC actions in terms of classical fields can now in principle be carried out by expanding the \star -products and the noncommutative fields by iterative use of the relations (3.10) and (3.11). The resulting formulae, already at second order in θ , become quite long, see for example the θ^2 formulae for the Yang-Mills action in [25] and the gravity action in [4]. They are not manifestly gauge invariant, and only repeated integrations by parts allow to express the actions in terms of gauge covariant quantities.

A better strategy leads directly to an explicit gauge invariant θ -expanded action: rather than expanding the \star -products and the elementary fields present in the NC action, we first study the SW map for composite fields (i.e., \star -products of fields and their derivatives), and find recursive relations in the noncommutativity parameter θ as well as in the number of fields. A special case of composite field is the action itself, and in the next section we give recursive and explicitly gauge invariant relations for the NC gravity action. The calculation of the expanded action at order θ^2 then becomes straightforward.

In this section and in Section 4, for the sake of notational simplicity, we omit the hat that denotes noncommutative fields, we also omit to write the \star and \wedge_\star products and simply write $\{ , \}$, $[,]$ rather than $\{ , \}_\star$, $[,]_\star$.

Lemma 1 Let P, Q be arbitrary exterior forms in the adjoint representation. Then

$$\{\Omega_{[A}, \mathbb{L}_{B]} P\} Q + P \{\Omega_{[A}, \mathbb{L}_{B]} Q\} + 2\ell_{[A} P \ell_{B]} Q = \{\Omega_{[A}, \mathbb{L}_{B]} (PQ)\} + 2L_{[A} P L_{B]} Q. \quad (3.13)$$

A similar formula holds also if Q transforms in the fundamental representation (simply omit the second and third curly brackets $\{ , \}$). Notice the algebraic character of this formula: it holds for any associative product between the symbols Ω_A, P, Q . The proof is by a straightforward calculation.

Lemma 2 Let

$$\frac{\partial P}{\partial \theta^{AB}} = \frac{i}{4} \{\Omega_{[A}, \mathbb{L}_{B]} P\} + P'_{[AB]}, \quad (3.14)$$

$$\frac{\partial Q}{\partial \theta^{AB}} = \frac{i}{4} \{\Omega_{[A}, \mathbb{L}_{B]} Q\} + Q'_{[AB]}, \quad (3.15)$$

where $P'_{[AB]}, Q'_{[AB]}$ are forms that characterize the deviation from the differential equation (3.6), (i.e., from $\frac{\partial P}{\partial \theta^{AB}} = \frac{i}{4} \{\Omega_{[A}, \mathbb{L}_{B]} P\}$). Then

$$\frac{\partial (PQ)}{\partial \theta^{AB}} = \frac{i}{4} \left(\{\Omega_{[A}, \mathbb{L}_{B]} (PQ)\} + 2L_{[A} P L_{B]} Q + P'_{[AB]} Q + P Q'_{[AB]} \right). \quad (3.16)$$

This result easily follows from the previous lemma and from the \star -product and \wedge_\star -product variation $P \wedge_{\star_{\theta+\delta\theta}} Q = P \wedge_{\star_\theta} Q + \frac{i}{2} \delta\theta^{AB} \ell_A P \wedge_\star \ell_B Q$.

Corollary 1 (Recursive relation for products of composite fields)

$$(PQ)^{n+1} = \frac{i \theta^{AB}}{4(n+1)} \left(\{\Omega_A, \mathbb{L}_B (PQ)\} + 2L_A P L_B Q + P'_{[AB]} Q + P Q'_{[AB]} \right)^n. \quad (3.17)$$

Proof By the same reasoning used to obtain the recursive solutions (3.10)-(3.12).

The next lemma relates the covariant Lie derivatives to the curvature tensor contracted along the commuting vector fields X_A and X_B , defined by¹

$$R_{AB} \equiv i_B i_A R . \quad (3.18)$$

The square of two exterior covariant derivatives equals the curvature tensor R ; similarly

Lemma 3 The commutator $[L_A, L_B] = L_A L_B - L_B L_A$ of two covariant Lie derivatives is equal to minus the curvature tensor R_{AB} :

$$[L_A, L_B] = -R_{AB} . \quad (3.19)$$

For any exterior form P that transforms in the adjoint we have

$$\theta^{AB} L_A L_B P = -\frac{1}{2} \theta^{AB} [R_{AB}, P] . \quad (3.20)$$

Another useful identity is

$$\theta^{AB} \mathbb{L}_A \Omega_B = \theta^{AB} R_{AB} . \quad (3.21)$$

Proof It is easy to verify (3.19) on fields ψ that transform in the fundamental representation $[L_A, L_B]\psi = -R_{AB}\psi$. The proof for fields in the adjoint is equivalent to (3.20) and is also straightforward.

In order to prove (3.21) we calculate

$$\begin{aligned} R_{AB} &= i_B i_A R = i_B i_A (d\Omega - \Omega \wedge \Omega) = i_B (\ell_A \Omega - d\Omega_A - \Omega_A \Omega + \Omega \Omega_A) \\ &= \ell_A \Omega_B - \ell_B \Omega_A - \Omega_A \Omega_B + \Omega_B \Omega_A \\ &= L_A \Omega_B - \ell_B \Omega_A \end{aligned} \quad (3.22)$$

where we used the Cartan formula $\ell_A = i_A d + d i_A$ and, in the second line, that $i_B \ell_A = \ell_A i_B$ because the vector fields X_A and X_B commute, and then again the Cartan formula. Formula (3.21) now follows immediately from the definition $\mathbb{L}_A = L_A + \ell_A$.

Corollary 2

$$\theta^{AB} \int Tr \left(\{ \Omega_A, \mathbb{L}_B(PQ) \} + 2 L_A P L_B Q \right) = \theta^{AB} \int Tr \left(\{ R_{AB}, P \} Q \right) . \quad (3.23)$$

The proof easily follows integrating by parts and using the cyclic property of $\int Tr$.

We now consider P and Q composite fields dependent on the noncommutative connection Ω and on the matter fields Φ .

¹Notice that if there exist local coordinates where the commuting vector fields X_A equal the partial derivatives ∂_μ , then we have $i_\nu i_\mu R = R_{\mu\nu}$, in accordance with $R = \frac{1}{2} R_{\mu\nu} dx^\mu \wedge dx^\nu$.

Using the previous results, and recalling that the contraction operators and the Lie derivative along commuting vector fields commute, we find:

$$R^{n+1} = \frac{i\theta^{AB}}{4(n+1)} \left(\{\Omega_A, \mathbb{L}_B R\} - [R_A, R_B] \right)^n \quad (3.24)$$

$$(R\Phi\Phi)^{n+1} = \frac{i\theta^{AB}}{4(n+1)} \left(\{\Omega_A, \mathbb{L}_B(R\Phi\Phi)\} + 2L_A R L_B(\Phi\Phi) - [R_A, R_B]\Phi\Phi + 2R L_A \Phi L_B \Phi \right)^n \quad (3.25)$$

$$\begin{aligned} ((RR)_{AB}\Phi\Phi)^{n+1} = \frac{i\theta^{CD}}{4(n+1)} & \left(\{\Omega_C, \mathbb{L}_D((RR)_{AB}\Phi\Phi)\} + 2L_C(RR)_{AB} L_D(\Phi\Phi) \right. \\ & \left. + 2(L_C R L_D R)_{AB}\Phi\Phi - \{[R_C, R_D], R\}_{AB}\Phi\Phi + 2(RR)_{AB} L_C \Phi L_D \Phi \right)^n \end{aligned} \quad (3.26)$$

$$(L_A \Phi)^{n+1} = \frac{i\theta^{CD}}{4(n+1)} \left(\{\Omega_C, \mathbb{L}_D(L_A \Phi)\} + 2\{R_{AC}, L_D \Phi\} \right)^n \quad (3.27)$$

$$\begin{aligned} (R L_A \Phi L_B \Phi)^{n+1} = \frac{i\theta^{CD}}{4(n+1)} & \left(\{\Omega_C, \mathbb{L}_D(R L_A \Phi L_B \Phi)\} + 2L_C R L_D(L_A \Phi L_B \Phi) \right. \\ & \left. - [R_C, R_D] L_A \Phi L_B \Phi + 2R[\{R_{AC}, L_D \Phi\}, L_B \Phi] + 2R L_C L_A \Phi L_D L_B \Phi \right)^n \end{aligned} \quad (3.28)$$

$$(D\Phi)^{n+1} = \frac{i\theta^{CD}}{4(n+1)} \left(\{\Omega_C, \mathbb{L}_D(D\Phi)\} - 2\{R_C, L_D \Phi\} \right)^n \quad (3.29)$$

where we have defined $[P, Q] = PQ - QP$, $\{P, Q\} = PQ + QP$ and $P_{AB} = i_B i_A P$ for exterior forms P, Q . In deriving (3.27) we found it convenient to rewrite formula (3.10) as $\Omega^{n+1} = \frac{i\theta^{CD}}{4(n+1)} (\{\Omega_C, \mathbb{L}_D \Omega\} - \{\Omega_C, d\Omega_D\})^n$, that implies $\Omega_A^{n+1} = \frac{i\theta^{CD}}{4(n+1)} (\{\Omega_C, \mathbb{L}_D \Omega_A\} - \{\Omega_C, \ell_A \Omega_D\})^n$; then we used $\ell_A \mathbb{L}_C \Phi = -[\ell_A \Omega_C, \Phi] + \mathbb{L}_C \ell_A \Phi$.

4 Recursive relations for the gravity action

A particular composite field is also the Lagrangian itself, and we calculate here the SW recursive relation for the noncommutative vierbein gravity action (2.1). In this case we set $\Phi = V$. Then Corollary 2, formula (3.25) and the identity $(RR)_{AB} = \{R_{AB}, R\} - [R_A, R_B]$ lead to the recursive relation

$$S^{n+2} = \frac{-\theta^{AB}}{4(n+2)} \int Tr \left(\gamma_5 ((RR)_{AB} VV + 2R L_A V L_B V) \right)^{n+1} \quad (4.30)$$

where, as in Section 3.1, we omit writing wedge and star products, and hats on noncommutative fields.

We now use (3.26) and (3.28) and further obtain

$$\begin{aligned} S^{n+2} = \frac{i\theta^{AB}\theta^{CD}}{4(n+2)4(n+1)} \int Tr & \left(\gamma_5 (\{[R_C, R_D], R\}_{AB} - \{R_{CD}, (RR)_{AB}\} - 2(L_C R L_D R)_{AB}) VV \right. \\ & \left. - 4\gamma_5 ((RR)_{AB} L_C V L_D V - R[\{R_{AC}, L_B V\}, L_D V] + R(L_A L_C V)(L_B L_D V)) \right)^n. \end{aligned} \quad (4.31)$$

Other equivalent expressions for S^{n+2} can be obtained by using the identities:

$$\begin{aligned} (RR)_{CD} &= \{R_{CD}, R\} - [R_C, R_D] , \\ \{\{R_{AB}, R\}, R\}_{CD} - 2\{R_{CD}, (RR)_{AB}\} &= 2(R R_{AB} R)_{CD} - \{(RR)_{AB}, R_{CD}\} . \end{aligned}$$

Note The general recursive relation for composite fields (3.17) follows from the corresponding differential equation (3.16). Similarly the recursive relations (3.24)-(4.30) are also implied by corresponding differential equations. In particular the action satisfies:

$$\frac{\partial}{\partial \theta^{AB}} S = -\frac{1}{4} \int Tr(\gamma_5((RR)_{AB} VV + 2R L_A V L_B V)) . \quad (4.32)$$

5 Extended gravity action at θ^2

If we set $n = 0$ in (4.31) we obtain the expression for S^2 where only commutative fields and usual undeformed products appear. Since in [4] we have shown that the expansion of the noncommutative gravity action in the commutative vierbein and spin connection fields contains only even powers of θ , then S^2 is the first nontrivial term in this expansion:

$$\int Tr(i\gamma_5 \hat{R} \wedge_\star \hat{V} \wedge_\star \hat{V}) = \int Tr(i\gamma_5 R \wedge V \wedge V) + S^2 + O(\theta^4) . \quad (5.1)$$

Finally, inserting the classical gamma expansions (2.7) of V and R and taking the trace brings the second order correction to the explicit form (repeated Lorentz indices are summed with the Minkowski metric η_{ab} ; the wedge product between forms is omitted):

$$\begin{aligned} S^2 &= \frac{1}{32} \theta^{AB} \theta^{CD} \int -\frac{1}{2} ((R_C^{ab} R_D^{cd} R^{ef})_{AB} - \frac{1}{2} R_{CD}^{ef} (R^{ab} R^{cd})_{AB}) V^g V^h (\varepsilon_{abcd} \delta_{ef}^{gh} + \varepsilon_{efgh} \delta_{ab}^{cd}) \\ &\quad + (2(L_C R^{ea} L_D R^{eb})_{AB} V^c V^d - (R^{ab} R^{cd})_{AB} L_C V^e L_D V^e \\ &\quad - 8R^{df} R_{AC}^{ab} L_B V^c L_D V^f - 4R^{ab} (L_A L_C V^c) (L_B L_D V^d)) \varepsilon_{abcd} . \end{aligned} \quad (5.2)$$

We conclude by observing that it is also possible to expand the noncommutative cosmological term $\frac{-i}{4!} \int Tr(\gamma_5 \Lambda_0 \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V})$ in power series of θ . Following the study in [4] on charge conjugation symmetry and parity under $\theta \rightarrow -\theta$, we again have that only even powers of θ contribute. We find:

$$\frac{-i}{4!} \int Tr(\gamma_5 \Lambda_0 \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V} \wedge_\star \hat{V}) = \frac{-i}{4!} \int Tr(\gamma_5 \Lambda_0 V \wedge V \wedge V \wedge V) + S_{\Lambda_0}^2 + O(\theta^4) \quad (5.3)$$

where

$$\begin{aligned} S_{\Lambda_0}^2 &= \frac{i\Lambda_0 \theta^{AB} \theta^{CD}}{4! 4} \int Tr \left(\gamma_5 \left(\left(\frac{1}{2} R_{AB} R_{CD} - R_{AC} R_{BD} \right) VVVV + \frac{1}{2} R_{AB} L_C (VV) L_D (VV) \right. \right. \\ &\quad \left. \left. + R_{AB} \{VV, L_C V L_D V\} + L_A V L_B V L_C V L_D V \right. \right. \\ &\quad \left. \left. - VV[\{R_{AC}, L_B V\}, L_D V] + VV(L_A L_C V)(L_B L_D V) \right) \right) . \end{aligned} \quad (5.4)$$

This expression agrees with the result in [22], Section 6, where the expansion of the NC cosmological term is obtained as the $\varphi \rightarrow 1$ limit of the θ -expansion of a term $\int Tr(i\gamma_5 \varphi VVVV)$ describing scalars φ in a curved background.

Acknowledgements

M. D. thanks the INFN sezione di Torino, gruppo collegato di Alessandria, for hospitality during her visit. The work of M.D. is supported by Project No.171031 of the Serbian Ministry of Education and Science.

A Cartan formulae

The usual Cartan calculus formulae simplify if we consider commuting vector fields X_A , and read

$$\ell_A = i_A d + di_A, \quad L_A = i_A D + Di_A \quad (\text{A.1})$$

$$[\ell_A, \ell_B] = 0, \quad [L_A, L_B] = i_A i_B R \quad (\text{A.2})$$

$$[\ell_A, i_B] = 0, \quad [L_A, i_B] = 0 \quad (\text{A.3})$$

$$i_A i_B + i_B i_A = 0, \quad d \circ d = 0, \quad D \circ D = R \quad (\text{A.4})$$

B Gamma matrices in $D = 4$

We summarize in this Appendix our gamma matrix conventions in $D = 4$.

$$\eta_{ab} = (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \quad (\text{B.1})$$

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \quad (\text{B.2})$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger = \gamma_5 \quad (\text{B.3})$$

$$\gamma_a^T = -C\gamma_a C^{-1}, \quad \gamma_5^T = C\gamma_5 C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C \quad (\text{B.4})$$

B.1 Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \quad (\text{B.5})$$

$$\gamma_{ab}\gamma_5 = \frac{i}{2}\varepsilon_{abcd}\gamma^{cd} \quad (\text{B.6})$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{B.7})$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{B.8})$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \quad (\text{B.9})$$

$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}_{cd}\gamma_5 - 4\delta_{[c}^{[a}\gamma_{d]}^{b]} - 2\delta_{cd}^{ab} \quad (\text{B.10})$$

$$Tr(\gamma_a\gamma^{bc}\gamma_d) = 8\delta_{ad}^{bc} \quad (\text{B.11})$$

$$Tr(\gamma_5\gamma_a\gamma_{bc}\gamma_d) = -4i\varepsilon_{abcd} \quad (\text{B.12})$$

where $\delta_{cd}^{ab} \equiv \frac{1}{2}(\delta_c^a \delta_d^b - \delta_c^b \delta_d^a)$, and indices antisymmetrization in square brackets has total weight 1.

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